Infinite products

$\prod_{n=1}^{\infty} p_n = \lim_{N\to\infty} \prod_{n=1}^{N} p_n$ converges if the limit is $\neq 0, \in \mathbb{C}$. We extend the notion to when also $p_{n_1} = \dots = p_{n_k} = 0$ and $\prod_{n \neq n_1,\dots,n_k} p_n$ converges in the sense above.

A necessary condition for convergence is $p_N = \frac{\prod_1^N p_n}{\prod_1^{N-1} p_n} \xrightarrow[N\to\infty]{} 1$

Comparison of $\underset{\textcircled{\Pi}}{\prod_{n=1}^{\infty}(1+a_n)}$ and $\underset{\textcircled{\Sigma}}{\sum_{n=1}^{\infty} \mathrm{Log}(1+a_n)}$ where $\mathrm{Log}(re^{it}) = \mathrm{Log}\, r + it, -\pi < t \leq \pi$.

<u>Lemma</u>. Π converges $\Leftrightarrow$ Σ converges.

<u>Pf</u>. ($\Leftarrow$) $(1+a_1)\cdots(1+a_N) = e^{\mathrm{Log}(1+a_1)+\dots+\mathrm{Log}(1+a_N)}$ so that Σ converges $\Rightarrow$ Π converges.

($\Rightarrow$) Let $\mathrm{Arg}(z) \in (-\pi,\pi]$, $\mathrm{Arg} = \mathrm{Im}(\mathrm{Log})$. Set $P_N = \prod_{n=1}^{N} p_n$. The implication

$P_N \to P_\infty \Rightarrow \sum_{n=1}^{N} \mathrm{Arg}(p_n) \to \mathrm{Arg}(P_\infty)$

fails: think of $p_n = e^{it_n}$ with $\sum_{n=1}^{\infty} t_n \notin (-\pi,\pi]$

$\frac{P_N}{P_\infty} \to 1 \Rightarrow \mathrm{Log}\left(\frac{P_N}{P_\infty}\right) \to 0$

and $\mathrm{Log}\left(\frac{P_N}{P_\infty}\right) = \mathrm{Log}\, e^{\mathrm{Log}(p_1)+\dots+\mathrm{Log}(p_N)-\mathrm{Log}(P_\infty)}$

$= \mathrm{Log}(p_1)+\dots+\mathrm{Log}(p_N) - \mathrm{Log}(P_\infty) + h_N \cdot 2\pi i$

with $h_N \in \mathbb{Z}$.

$\Rightarrow 2\pi i \cdot (h_{N+1} - h_N) = \mathrm{Log}\left(\frac{P_{N+1}}{P_\infty}\right) - \mathrm{Log}\left(\frac{P_N}{P_\infty}\right) - \mathrm{Log}(p_{N+1})$

$\xrightarrow[N\to\infty]{} 0 - 0 - 0$, i.e. $h_N = h_\infty \in \mathbb{Z}$ definit.

and $\mathrm{Log}(p_1)+\dots+\mathrm{Log}(p_N) = \mathrm{Log}(P_\infty) + \mathrm{Log}\left(\frac{P_N}{P_\infty}\right)$

$- h_N \cdot 2\pi i \xrightarrow[N\to\infty]{} \mathrm{Log}(P_\infty) - h_\infty \cdot 2\pi i$ ■

It would be tempting to ~~[illegible]~~ guess

$\underset{\text{converges}}{\sum \mathrm{Log}(1+a_n)} \Leftrightarrow \underset{\text{converges}}{\sum a_n}$,

but both implications fail.

($\not\Leftarrow$) $\mathrm{Log}(1+a_n) = a_n - \frac{a_n^2}{2} + O(a_n^3)$

and if $a_n = \frac{(-1)^n}{\sqrt{n}}$, $\sum a_n$ converges

$\sum \mathrm{Log}(1+a_n)$ does not

($\not\Rightarrow$) [illegible]

[illegible] If $\mathrm{Log}(1+a_n) = b_n = \frac{(-1)^n}{\sqrt{n}}$

then $\sum \mathrm{Log}(1+a_n)$ converges, but

$\sum a_n = \sum(e^{b_n} - 1) = \sum\left(b_n + \frac{b_n^2}{2} + O(b_n^3)\right) = \sum b_n + \frac{1}{2}\sum\frac{1}{n}$

$+ \sum O(b_n^3)$ diverg.

Lemma $\Omega \subseteq \mathbb{C}$ open, s.c., $f \in Hol(\Omega)$, $f(z) \neq 0$ in Ω

$\Rightarrow \exists g \in Hol(\Omega)$ s.t. $f = e^{g}$.

Pf. $\frac{f'}{f} \in Hol(\Omega)$, Ω s.c. $\Rightarrow \exists g \in Hol(\Omega): g' = \frac{f'}{f}$

Hence, $(e^{-g} f)'(z) = e^{-g}(-g'f + f') = 0$

$\Rightarrow e^{-g(z)} f(z) = \text{const} \neq 0$ ■

Corollary. If $f \in Hol(\mathbb{C})$ is entire with zeros at 0 (multiplicity $m \geq 0$), $a_1, \ldots, a_N \in \mathbb{C} \setminus \{0\}$, then

$$f(z) = z^m \cdot \prod_{n=1}^{N} \left(1 - \frac{z}{a_n}\right) e^{g(z)} \quad \exists g \in Hol(\mathbb{C}).$$

Wish: $f(z) = z^m \cdot \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \cdot e^{g(z)}$

in general. But this requires convergence of $\prod_{n=1}^{\infty}\left(1 - \frac{z}{a_n}\right)$, uniformly on compact sets.

We need more info on products.

Def. $\prod_{n=1}^{\infty}(1+a_n)$ converges absolutely $\overset{\text{def}}{\Longleftrightarrow}$ $\sum_{n=1}^{\infty} |Log(1+a_n)| < +\infty$.

From calculus: $\sum_{n=1}^{\infty} |Log(1+a_n)| < +\infty \Leftrightarrow \sum_{n=1}^{\infty} |a_n| < +\infty$.

Sequences of holomorphic functions.

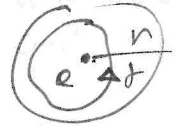
Thm (Weierstrass). If $f_n \in Hol(\Omega_n)$ and $f_n \to (f : \Omega \to \mathbb{C})$ uniformly on compact sets, then

(a) $f \in Hol(\Omega)$; (b) $f_n' \to f'$ uniformly on compact sets.

Def. $f_n \to f$ uniformly on compact sets means that

$\forall K \subset\subset \Omega \Rightarrow \begin{cases} K \in \Omega_n \text{ definitely on } n \\ \text{and } f_n \to f \text{ unif. on } \Omega. \end{cases}$

Pf. Use Morera's Theorem.



Let γ be a closed curve in $\overline{\Delta(a,r)} = \{z : |z-a| \leq r\}$. Then,

$$\int_\gamma f(z)\,dz = \lim_n \int_\gamma f_n(z)\,dz = 0 \Rightarrow f \in Hol(\overline{\Delta(a,r)})$$

Also:

$$|f'(z) - f_n'(z)| = \left|\frac{1}{2\pi i} \int_{\partial\Delta(a,r)} \frac{f(w) - f_n(w)}{(w-z)^2}\,dw\right|$$

$$\leq \sup_{|w-a|=r} |f(w) - f_n(w)| \cdot \frac{2\pi r}{2\pi} \frac{1}{(r/2)^2} \quad \text{if } |z-a| \leq r/2$$

$\xrightarrow[n\to\infty]{} 0$ for each fixed $\overline{\Delta(a,r)} \subset \Omega$. ■

Theorem (Hurwitz). If $f_n(z) \neq 0$ in Ω, $f_n \in Hol(\Omega)$, and $f_n \to f$ uniformly ~~on~~ ration compact sets, then either $f \equiv 0$ in Ω or $f(z) \neq 0 \; \forall z \in \Omega$

Proof. Suppose $f \not\equiv 0$, $f(z_0)=0$ and $f(z) \neq 0$ in $0<|z-z_0|\leq r$, so that

$$\left|\frac{1}{f_n(z)} - \frac{1}{f(z)}\right| = \frac{|f(z)-f_n(z)|}{|f_n(z)|\cdot|f(z)|} \leq \frac{|f_n(z)-f(z)|}{\min\limits_{|w-z_0|=r} |f(w)|^2/2}$$

if $|z-z_0|=r$ and $n \geq n_0$,

$\xrightarrow[n\to\infty]{} 0$ uniformly.

Also, by Weierstrass, $f_n'(z) \longrightarrow f'(z)$ uniformly on $|z-z_0|=r$

so $$\lim_{n\to\infty} \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f_n'(z)}{f_n(z)}\,dz \overset{(*)}{=} \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f'(z)}{f(z)}\,dz$$

||

$\lim\limits_{n\to\infty}$ # {roots of $f_n=0$ inside $|z-z_0|=r$}

$\Rightarrow$ # {roots of $f=0$ inside $|z-z_0|=r$} $=0$, contradiction ∎

$$(*)\quad \frac{f_n'(z)}{f_n(z)} - \frac{f'(z)}{f(z)} = \left(\frac{f_n'(z)}{f_n(z)} - \frac{f_n'(z)}{f(z)}\right) + \left(\frac{f_n'(z)}{f(z)} - \frac{f'(z)}{f(z)}\right).$$

__Applications to infinite products on a region Ω.__

Suppose $\prod_{n=1}^{\infty} f_n(z) = p(z)$ converges u.c.

Then, either $p\equiv 0$ or $\{z : p(z)=0\} = \bigcup_n \{z : f_n(z)=0\}$.

Pf. $\{z : p(z)=0\} \supseteq \bigcup_n \{z : f_n(z)=0\}$.

If R.H.S. has accumulation points in $\Omega \Rightarrow p\equiv 0$.

If not, $\Omega' = \Omega \setminus$ R.H.S. is a region satisfying the Hp. of Hurwitz $\Rightarrow$ either $p\equiv 0$, or $p(z)\neq 0$ for $z\in\Omega'$. i.e.

L.H.S. $\subseteq$ R.H.S. ∎

__Weierstrass factorization.__ Pick $a_n \xrightarrow[n]{} \infty$ in $\mathbb{C}$

We would like to define

$$P(z) = \prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)$$

and prove it converges u.c.

Convergence is absolute in $|z|\leq R$ if

$\infty > \sum_n \frac{R}{|a_n|}$ i.e. if $\sum_n \frac{1}{|a_n|} < \infty$

(and in fact it's uniform as well).

What about the general case?

The idea in enhancing convergence by replacing $1-\frac{z}{a_n}$ with $\left(1-\frac{z}{a_n}\right)e^{P_n(z)}$, where P_n is related to the Taylor expansion of $\mathrm{Log}\left(1-\frac{z}{a_n}\right)$.

$$\mathrm{Log}(1-w) = w + \frac{w^2}{2} + \cdots + \frac{w^m}{m} + \rho_m(w)$$

$$\mathrm{Log}(1-w) - \left(w + w^2/2 + \cdots + w^m/m\right) = \rho_m(w)$$

Set $p_n(z) = -\frac{z}{a_n} - \frac{1}{2}\left(\frac{z}{a_n}\right)^2 - \dots - \frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}$

with m_n to be chosen, and let $r_n(z) = \rho_{m_n}(z/a_n)$.

Fix $R > 0$ and consider $n(R)$ s.t. $|a_n| > R$ if $n \geq n(R)$.

Then

$$\operatorname{Log}\left(1 - \frac{z}{a_n}\right) + p_n(z) = r_n(z)$$

$$= -\frac{1}{m_n+1}\left(\frac{z}{a_n}\right)^{m_n+1} + \dots$$

so $|r_n(z)| \leq \frac{(R/|a_n|)^{m_n+1}}{m_n+1} \cdot \frac{1}{1 - R/|a_n|}$

~~we~~ and so we want

$$(*)\ \infty > \sum_n \frac{(R/|a_n|)^{m_n+1}}{m_n+1} \geq \sum_n |r_n(z)|$$

Obs. that in this case $|r_n(z)| \to 0$, hence $|\arg(r_n(z))| \leq \pi$ definit., so that $r_n(z) = \operatorname{Log}\left(e^{r_n(z)}\right)$

In order to have $(*)$ it suffices to set $m_n = n^2$.

Let's run the argument backward.

<u>Lemma</u> Let $a_n \to \infty$ and set

$$p_n(z) = -\frac{z}{a_n} - \frac{1}{2}\left(\frac{z}{a_n}\right)^2 - \dots - \frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}.$$

If the m_n's are chosen in such a way that

$$\sum_n \frac{(R/|a_n|)^{m_n+1}}{m_n+1} < +\infty \quad \forall R > 0,$$

then

$$\prod_{n=1}^{\infty}\left(1 - \frac{z}{a_n}\right) e^{p_n(z)} = h(z),$$

converges absolutely and uniformly on $\overline{\Delta(0,R)}$ $\forall R > 0$ (hence, u.c.).

<u>Factorization Theorem (Weierstrass).</u>

Let $f \in \operatorname{Hol}(\mathbb{C})$. Then f can be written

$$f(z) = z^m \prod_{n=1}^{\infty}\left(1 - \frac{z}{a_n}\right) e^{p_n(z)} \cdot e^{g(z)}$$

with $g \in \operatorname{Hol}(\mathbb{C})$, $m \geq 0$.

<u>Pf.</u> Let $\{a_n\}$ be the sequence of zeros of f, $a_n \neq 0$, and m the order of zero at $z = 0$.

Then, $\frac{f(z)}{z^m \cdot h(z)} = e^{g(z)}$ because it is zero-free ■

~~Corollary of the proo~~ In the same spirit, we have a theorem concerning meromorphic function.

Recall that f is meromorphic in a region $\Omega \subseteq \mathbb{C}$, $f \in \mathrm{Mer}(\Omega)$, if it is holomorphic in $\Omega \setminus \{b_n\}$ and at each b_n: $f(z) = \frac{a_{-m_n}}{(z-b_n)^{m_n}} + \dots + \frac{a_{-1}}{z-b_n} + h_n(z)$

with h_n holomorphic in $\Delta(b_n, r_n)$, $r_n > 0$. This is equivalent to asking that $\forall b \in \Omega$ $\exists m \geq 0$: $\lim_{z \to b} (z-b)^m f(z) \in \mathbb{C}$.

If $f(z) = \frac{a_{-m}}{(z-b)^m} + \dots + \frac{a_{-1}}{z-b} + h(z)$

with h holomorphic in a neighb. of b, $Q\left(\frac{1}{z-b}\right) = \frac{a_{-m}}{(z-b)^m} + \dots + \frac{a_{-1}}{z-b}$ is the <u>principal</u> <u>part</u> of f at b.

<u>Theorem</u> [Mittag-Leffler for $\Omega = \mathbb{C}$]. Let $\{b_n\} \subseteq \mathbb{C}$, $\lim b_n = \infty$, and for each n consider a polynomial $Q_n(w) = a^{(n)}_{-m_n} w^{m_n} + \dots + a^{(n)}_{-1}(w)$.

Then, $\exists f \in \mathrm{Mer}(\mathbb{C})$ having ~~pole~~ principal part $Q_n\left(\frac{1}{z-b_n}\right)$ at b_n and no other poles.

Moreover, all such f's have the form:

$$f(z) = \sum_n \left[Q_n\left(\frac{1}{z-b_n}\right) - q_n(z)\right] + H(z)$$

where $q_n(z)$ are suitable polynomials, and H is entire.

- If the sequence is finite, we can take $q_n = 0$.

<u>Proof.</u> The case of finitely many points is obvious (like it was that of finitely many zeros). We also may assume that $b_n \neq 0$ $\forall n$.

~~If Q is the polyniel, $b \neq 0$, and P_m is the Taylor polynomiel of $Q\left(\frac{1}{z-b}\right)$ having degree m, and~~ .

We need an estimate on the remainder term in Taylor's formula.

Lemma. Let $f \in Hol(\Omega)$, $\Omega \ni a$,

$$(*)\ f(z) = f(a) + f'(a)(z-a) + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(z-a)^{n-1} + f_n(z)\cdot(z-a)^n.$$

Then, if C_1 is a circle centered at a, contained in Ω with its interior, and $z \neq a$ lies inside C_1,

$$f_n(z) = \frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{(w-a)^n(w-z)}\,dw.$$

Pf. $\cancel{\frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{(w-a)^n(w-z)}\,dw}$

$$*\ f_n(z) = \frac{1}{2\pi i}\int_{C_1}\frac{f_n(w)}{w-z}\,dw$$

$$\overset{(*)}{=}\frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{(w-a)^n(w-z)}\,dw - \sum_{l=0}^{n-1}\frac{f^{(l)}(a)}{l!}\frac{1}{2\pi i}\int_{C_1}\frac{(w-a)^l\,dw}{(w-a)^n(w-z)}$$

It suffices to show that $\frac{1}{2\pi i}\int_{C_1} G_m(w)\,dw = 0$

where $G_m(w) = \frac{1}{(w-a)^m(w-z)}$, $m \geq 1$

$$\frac{1}{2\pi i}\int_{C_1} G_m(w)\,dw = Res(G_m(w), w=z) + Res(G_m(w), w=a).$$

$$Res(G_m(w), w=z) = \lim_{w \to z}(w-z)\,G_m(w) = \frac{1}{(z-a)^m}$$

and

$$G_m(w) = \frac{1}{(w-a)^m\cdot[(w-a)-(z-a)]} =$$

$$= -\frac{1}{z-a}\cdot\frac{1}{(w-a)^m}\cdot\frac{1}{1-\frac{w-a}{z-a}}$$

$$\underset{|w-a|<|z-a|}{\overset{\uparrow}{=}} -\frac{1}{z-a}\,\frac{1}{(w-a)^m}\cdot\sum_{l=0}^{\infty}\frac{(w-a)^l}{(z-a)^l}$$

$$= \frac{1}{w-a}\left\{-\frac{1}{z-a}\cdot\frac{1}{(z-a)^{m-1}}\right\} + \text{other terms}$$

$$\Rightarrow Res(G_m(w), w=a) = -\frac{1}{(z-a)^m} \ ∎$$

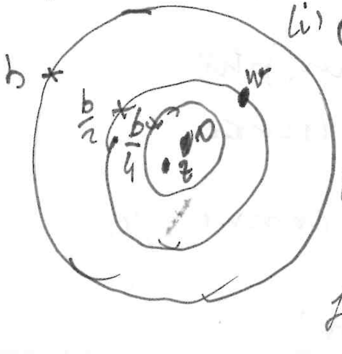
Back to the proof.

WLOG we can assume that all $b_n \neq 0$.

Let $Q = Q_n$ and $q = q_n$ be the Taylor polynomial of order $N = N_n$ at $z=0$ of $\left(Q\left(\frac{1}{z-b}\right), b = b_n\right)$

By the Lemma,

$$(i)\ Q\left(\frac{1}{z-b}\right) - q(z) = z^{N+1}\cdot f_{N+1}(z)$$



where

$$(ii)\ f_{N+1}(z) = \frac{1}{2\pi i}\int_{|w|=\frac{|b|}{2}}\frac{Q\left(\frac{1}{w-b}\right)dw}{w^{N+1}\cdot(w-z)}$$

for $|z| < |b|/2$.

Let $M = M_n = \max_{|z| = \frac{|b|}{2}} \left| Q\left(\frac{1}{z-b}\right) \right|$.

By (i) and (ii),

$$\left| Q\left(\frac{1}{z-b}\right) - q(z) \right| \leq \frac{|z|^{\nu+1} \cdot M}{2\pi} \cdot 2\pi \frac{|b|}{2} \cdot \left(\frac{|b|}{2}\right)^{-(\nu+1)} \left(\frac{|b|}{4}\right)^{-1}$$

$$= 2M \cdot \left(\frac{2|z|}{|b|}\right)^{\nu+1}.$$

Having this estimate at hand $\forall n \geq 1$ we can finish the proof. We want

$$\sum_{n=1}^{\infty} 2M_n \left(\frac{2R}{|b_n|}\right)^{\nu_n+1} \text{ to converge } \forall R \geq 0$$

i.e. $\limsup_{n \to \infty} \frac{|b_n|}{M_n^{1/\nu_n+1}} = +\infty$

This can be achieved by choosing ν_n large enough, since $|b_n| \to \infty$.

Consider now $R > 0$. The series

$$\sum_{n:\, \frac{|b_n|}{4} > R} \left[Q_n\left(\frac{1}{z-b_n}\right) - q_n(z) \right]$$

converges to a holomorphic function for $|z| \leq R$, hence

$$(*) \quad \sum_{n=1}^{\infty} \left[Q_n\left(\frac{1}{z-b_n}\right) - q_n(z) \right] \text{ converges}$$

to a function h which is holomorphic in $|z| \leq R$ but for poles at $z = b_n$ $(|b_n| \leq R)$ where the principal part is $Q_n\left(\frac{1}{z-b_n}\right)$.

Thus, $(*)$ converges to a function which is meromorphic in $\mathbb{C}$, with poles at $z = b_n$ and the given principle parts there ▫

Theorem. Let $\{b_n\}$ be a sequence in $\mathbb{C}$ s.t. $|b_n| \to \infty$ and let $\{a_n\}$ be a sequence of target values.

Then, $\exists f \in Hol(\mathbb{C})$ (entire) s.t. $f(b_n) = a_n$.

Pf. By Weierstrass Theorem, we can find $g \in Hol(\mathbb{C})$ s.t. $g(b_n) = 0$ $\forall n \geq 1$ (simple zeros). By Mittag-Leffler we find $h \in Mer(\mathbb{C})$ with simple poles at each b_n and principal part $Q_n(z) = \frac{a_n}{g'(b_n)(z-b_n)}$.

Then, $f = g \cdot h \in Hol(\mathbb{C})$ and

$$f(b_n) = \lim_{z \to b_n} \frac{g(z)}{z-b_n} \cdot a_n \cdot \frac{1}{g'(b_n)} = a_n \ ▪$$